PHYS 321

Assignment 2

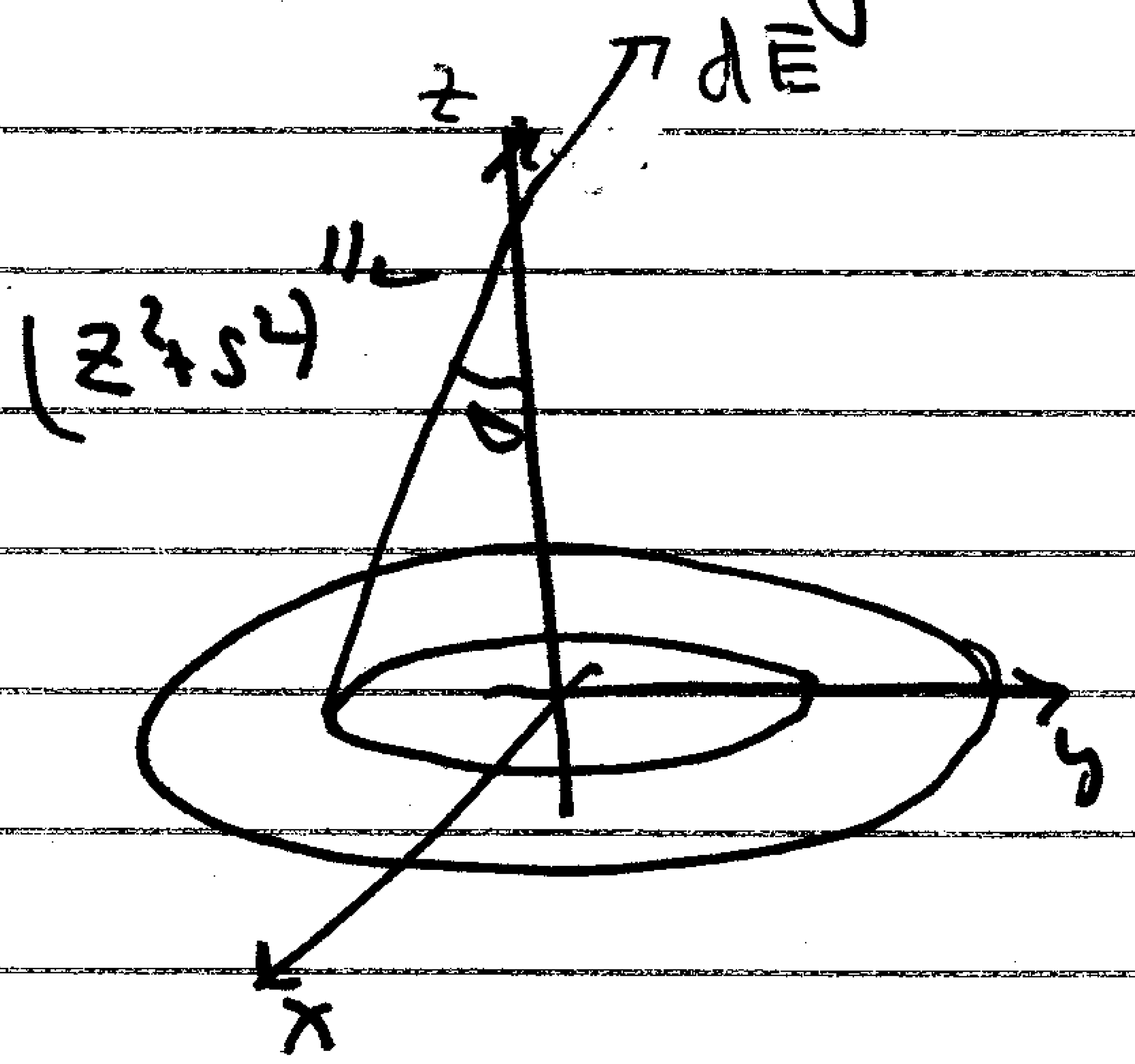
Due Monday Feb 26, 2018

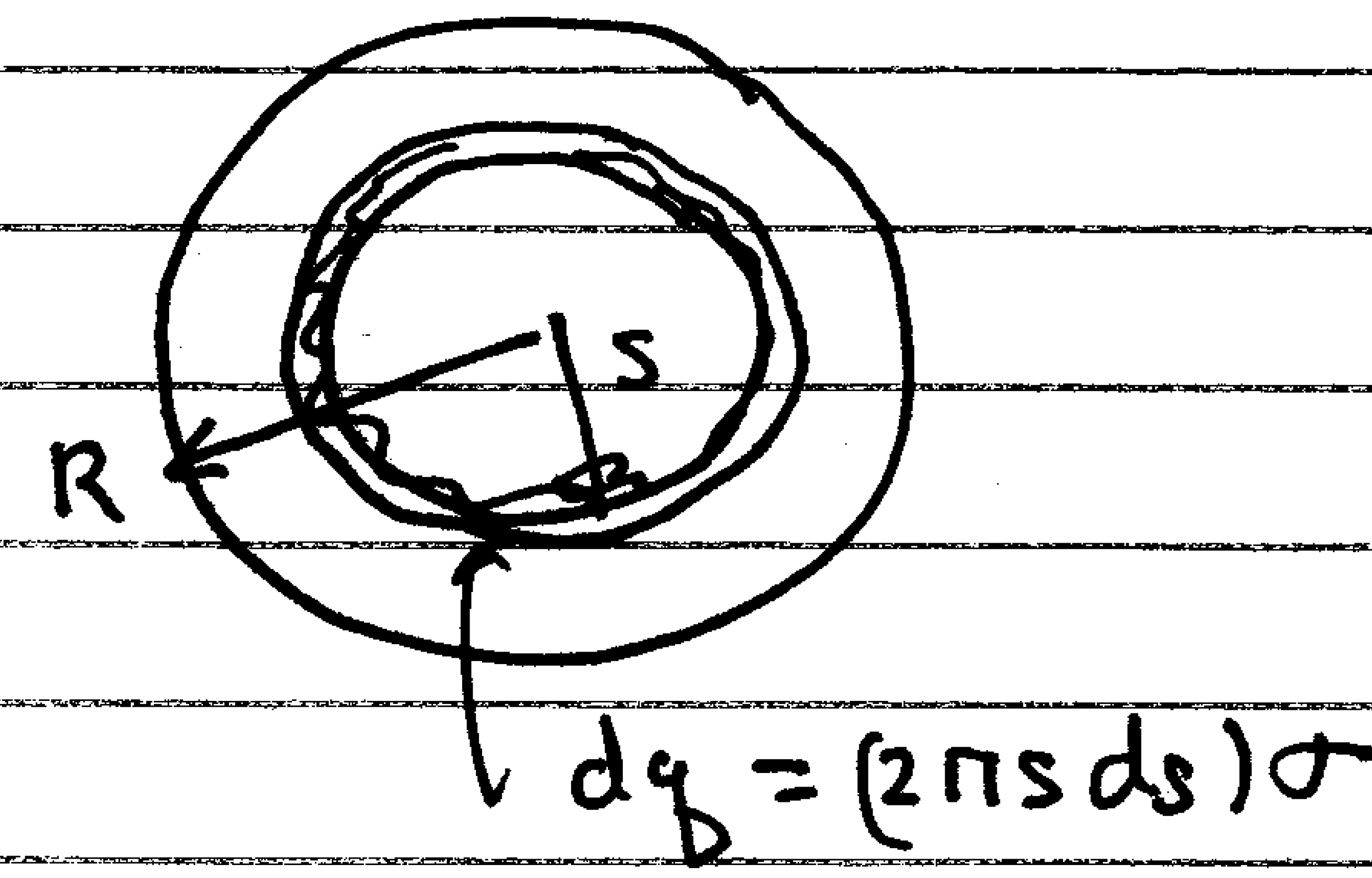
Read Chapter 2, Chapter 3.1, 3.3.2, 3.4

Problems: 2.6, 2.15, 2.20, 2.21, 2.25c, 2.34, 2.43, 3.28

SP 1: The upper hemisphere of a conducting spherical shell of radius "a" has potential V_o. The lower hemisphere has potential $-V_o$. The two hemispherical shells are separated by a thin non-conducting ring at the "equator". Find a solution for the potential $V(r, \theta)$ for all space. r, θ are the usual spherical co-ordinates. Choose as your co-ordinate origin the center of the sphere.

2.6 Divide the disc into differential rings.





$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(z^2+s^2)}(\cos\theta) \qquad E_x = E_y = 0 \text{ by symmetry}$$

$$\cos\theta = \frac{z}{(z^2+s^2)^{1/2}}$$

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{(2\pi s\,ds)\sigma\, z}{(z^2+s^2)^{3/2}}$$

$$E_z = \frac{\sigma z}{2\varepsilon_0}\int_0^R ds\,\frac{s}{(z^2+s^2)^{3/2}}$$

$$= \frac{\sigma z}{2\varepsilon_0}(-)(z^2+s^2)^{-1/2}\Big|_{s=0}^{s=R}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{(z^2+R^2)^{1/2}}\right)\hat{z}$$

$R \to \infty$ $E = \frac{\sigma}{2\varepsilon_0}$ as expected – infinite plane

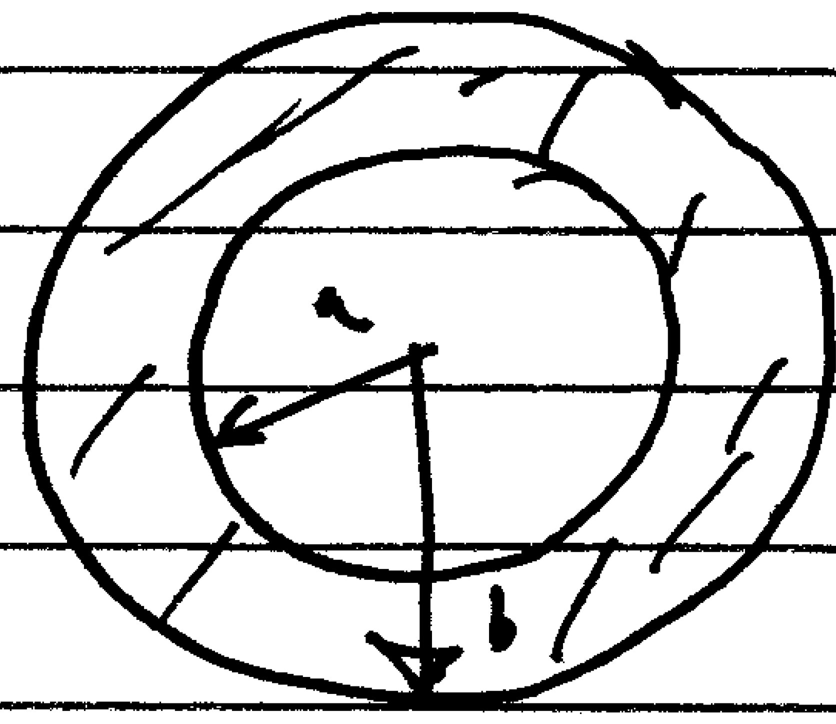
As $z \to \infty$ we have $\dfrac{z}{z\left(1+\dfrac{R^2}{z^2}\right)^{1/2}} \to 1 - \dfrac{1}{2}\dfrac{R^2}{z^2}$

$$\vec{E} \to \frac{\sigma}{2\varepsilon_0}\left(1 - 1 + \frac{1}{2}\frac{R^2}{z^2}\right) = \frac{\sigma}{2\varepsilon_0}\left(\frac{1}{2}\frac{R^2}{z^2}\right)$$

$$\sigma \pi R^2 = Q$$

$$E = \frac{Q}{4\pi R^2 \varepsilon_0}\frac{R^2}{z^2} = \frac{Q}{4\pi\varepsilon_0 z^2}$$ as expected, point-like charge.

2.15) $\rho = \frac{k}{r^2} \quad a \le r \le b$



a) $r < a \quad E = 0$

b) $a < r < b$ Apply Gauss

$$\oint \bar{E} \cdot d\bar{a} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_a^r \left(\frac{k}{r'^2}\right) 4\pi r'^2 dr'$$

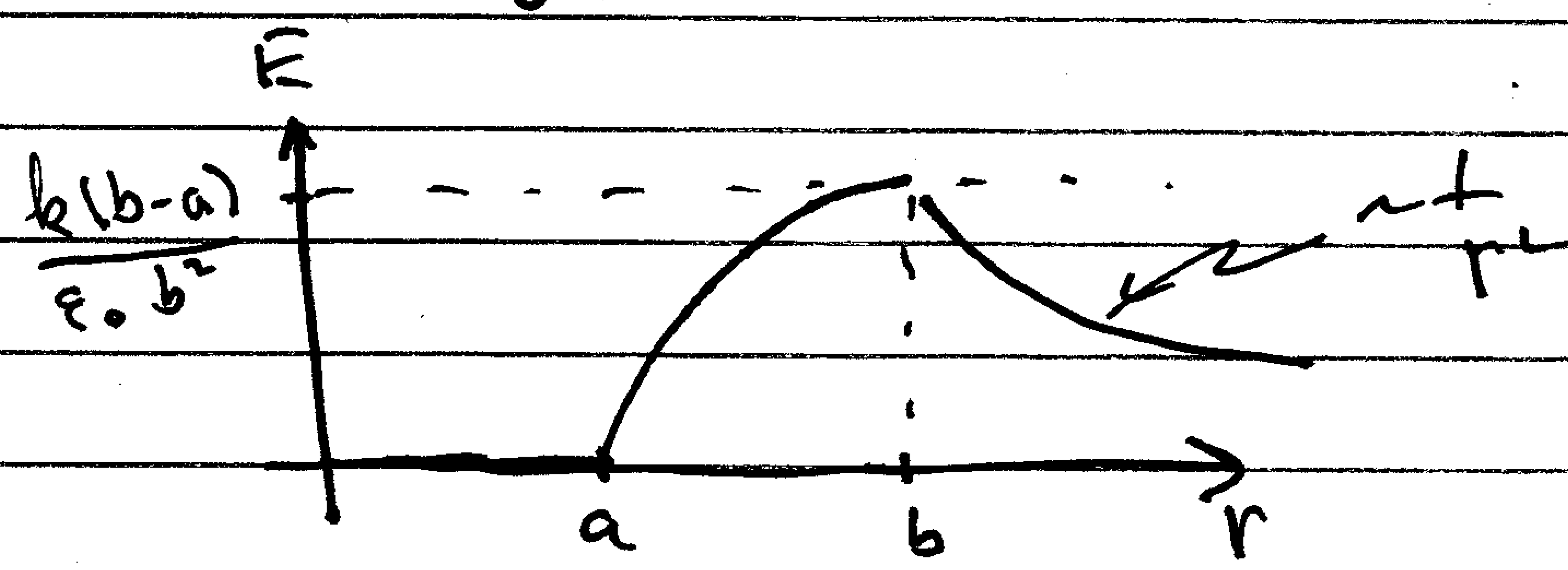
$$= \frac{1}{\varepsilon_0}(4\pi k) \int_0^r dr' = \frac{4\pi k}{\varepsilon_0}(r-a) = E(4\pi r^2)$$

$$E = \frac{k}{\varepsilon_0}\left(\frac{r-a}{r^2}\right)$$

c) $r > b \quad \oint \bar{E} \cdot d\bar{a} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_a^b \left(\frac{k}{r'^2}\right) 4\pi r'^2 dr'$

$$= \frac{4\pi k}{\varepsilon_0}(b-a) = (4\pi r^2) E$$

$$E = \frac{k(b-a)}{\varepsilon_0 r^2}$$



2.20

a) $\vec{E} = k\left(xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z}\right)$

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= k\left[\hat{x}\left(-2y\right) - \hat{y}\left(3z\right) + \hat{z}\left(-x\right)\right] \neq 0$$

must have $\nabla \times \vec{E} = 0$

b) $\vec{E} = k\left[y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + 2yz\,\hat{z}\right]$

$$\nabla \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

$$= k\left[\hat{x}\left(2z - 2z\right) - \hat{y}\left(0\right) + \hat{z}\left(2y - 2y\right)\right] = 0$$

this $\vec{E}$ is <u>OK</u>.

$$\bar{E}\cdot d\bar{l} = k\left(y^2 dx + (2xy + z^2)\,dy + 2yz\,dz \right)$$

Interate along x-axis to an arbitrary pt. But $y = 0$ along this path and $dy = dz = 0$

Now integrate along y-axes to an arbitrary pt. $z = 0$, $dx = dz = 0$

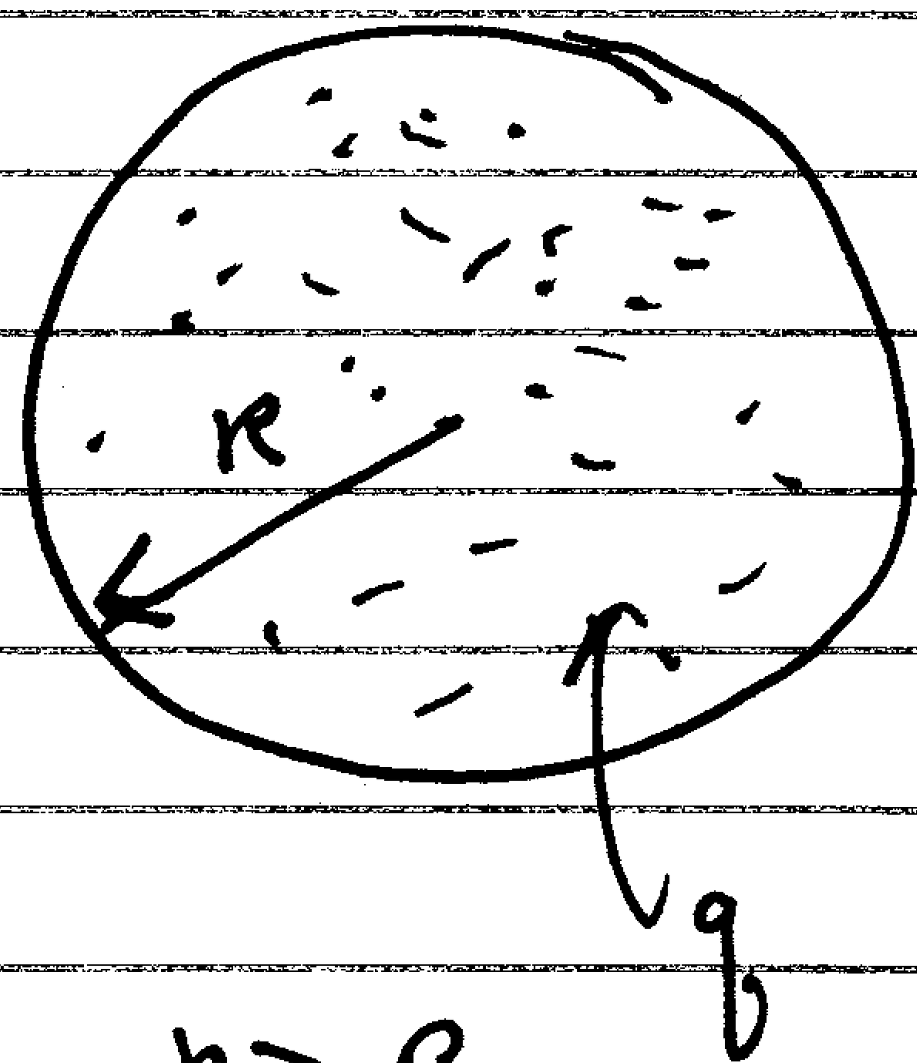
We get kxy^2 contribution.

Now integrate along z-axes to an arbitrary pt. $dx = dy = 0$

We get kyz^2 contribution

$$V(x,y,z) = -k\left(xy^2 + yz^2\right)$$

2.21 Outside $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$



$$V(r) = \frac{q}{4\pi\epsilon_0 r} \qquad \underline{r > R}$$

$$\text{Check } -\nabla V = -\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0 r}\right)\hat{r}$$

$\underline{r > R}$

$$= \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \quad \text{and } V(r\to\infty) = 0$$

Inside the sphere

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r} \qquad V(r) = -\frac{q}{4\pi\epsilon_0 R^3}\left(\frac{r^2}{2}\right) + \text{const}$$

We must $V(R) = \frac{q}{4\pi\epsilon_0 R}$ since V is continuous.

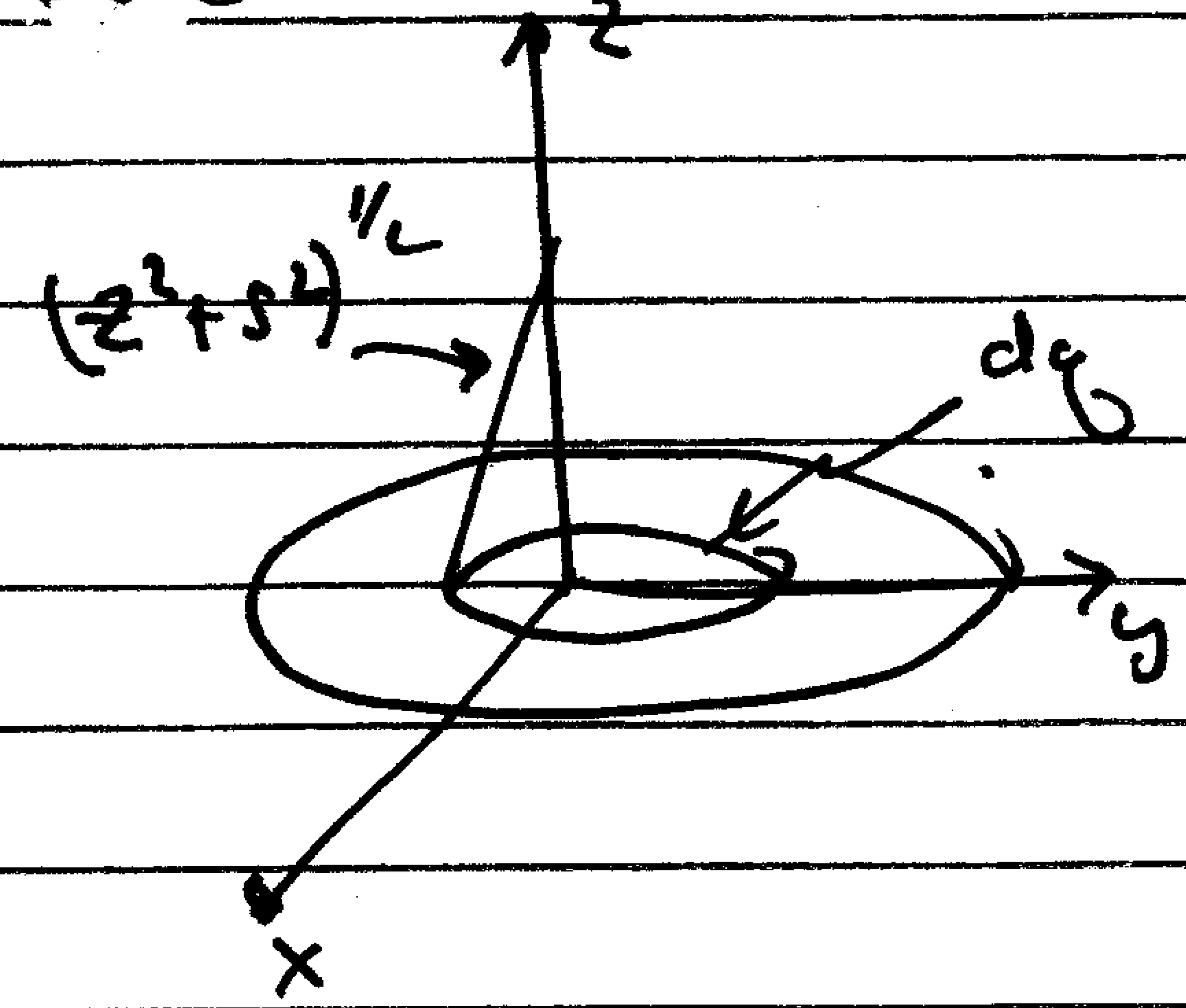
$$V(R) = -\frac{q}{4\pi\epsilon_0}\frac{R^2}{2R^3} + K = \frac{q}{4\pi\epsilon_0 R}$$

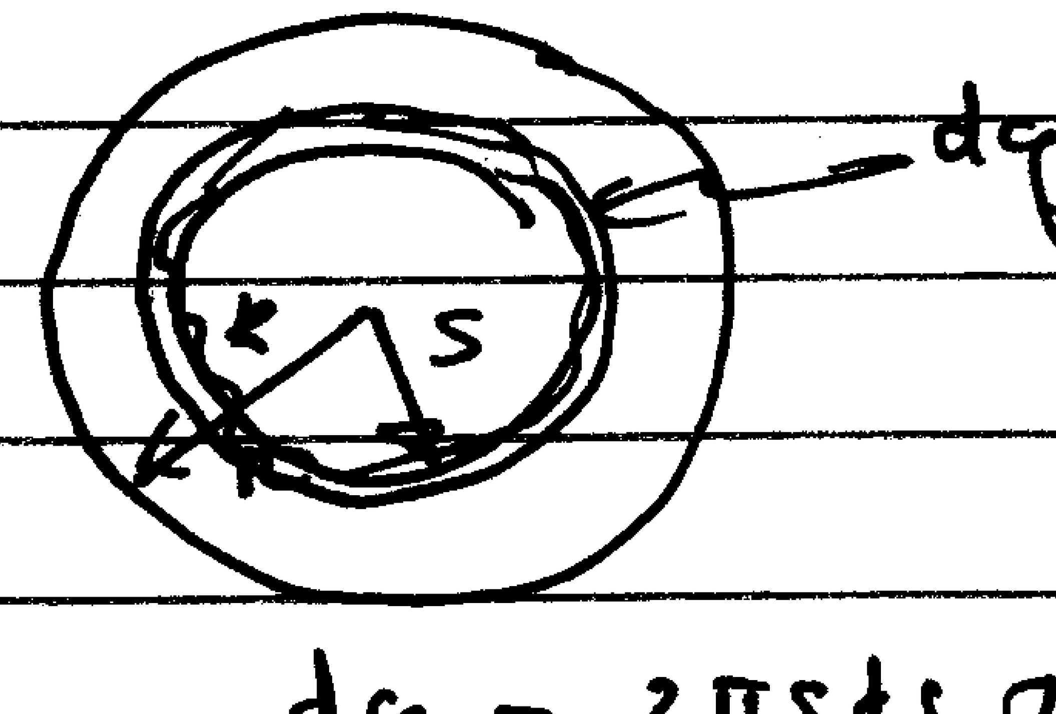
$$K = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{R^2}{2R^3}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{3}{2R}\right)$$

$$V(r) = +\frac{q}{4\pi\epsilon_0}\left(\frac{1}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad \underline{r < R}$$

2.25c

Divide the disc into differtial rings.





$$dq = 2\pi s\, ds\, \sigma$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(z^2+s^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi s\,\sigma\, ds}{(z^2+s^2)^{1/2}}$$

$$V(z) = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{s\,ds}{(z^2+s^2)^{1/2}} = \frac{\sigma}{2\varepsilon_0}(z^2+s^2)^{1/2}\Big|_{s=0}^{s=R}$$

$$V(z) = \frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2} - |z|\right]$$

Let $R\to\infty$

$$V(z) = \frac{\sigma}{2\varepsilon_0}\left[R\left(1+\frac{z^2}{R^2}\right)^{1/2} - z\right] \quad z>0$$

$$= \frac{\sigma}{2\varepsilon_0}(R-z) \qquad -\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0} = E_z \text{ as expected}$$

Let $z\to\infty$ $\quad V(z) = \frac{\sigma}{2\varepsilon_0}\left[z\left(1+\frac{R^2}{z^2}\right)^{1/2} - z\right]$

$$= \frac{\sigma}{2\varepsilon_0}\left[z + \frac{1}{2}\frac{R^2}{z} - z\right] = \left(\frac{q}{\pi R^2}\right)\left(\frac{1}{2\varepsilon_0}\right)\left(\frac{R^2}{2z}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 z}$$ – point charge result.

Generally we have

$$E(z) = -\frac{\partial V}{\partial z} \qquad V(z) = \frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2} - z\right] \quad z>0$$

$$E(z) = -\frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{-1/2}\left(\tfrac{1}{2}\right)(2z) - 1\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{(z^2+R^2)^{1/2}}\right]$$

Same result as prob 2.6.

2.34) a) $W = \frac{1}{2}\int \rho V \, d\tau$

$r < R \quad \rho = \frac{q}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{q}{R^3} \qquad \rho = 0 \quad r > R$

$V(r) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad r \leq R$

$W = \frac{1}{2}(4\pi)\int_0^R r^2 dr \left(\frac{3}{4\pi}\frac{q}{R^3}\right)\left(\frac{q}{4\pi\varepsilon_0}\frac{1}{2R}\right)\left(3 - \frac{r^2}{R^2}\right)$

$= \pi\left(\frac{3}{4\pi}\right)\left(\frac{q}{R^3}\right)\left(\frac{q}{4\pi\varepsilon_0}\right)\left(\frac{1}{2R}\right)\int_0^R dr\left(3r^2 - \frac{r^4}{R^2}\right)$

$= \frac{q^2}{4\pi\varepsilon_0}\left(\frac{3}{2}\right)\left(\frac{1}{2R^4}\right)\left(R^3 - \frac{R^3}{5}\right)$

$= \frac{q^2}{4\pi\varepsilon_0 R}\left(\frac{3}{4}\right)\left(\frac{4}{5}\right) = \frac{3}{5}\frac{q^2}{4\pi\varepsilon_0 R}$

b) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau$

$r < R \quad E = \frac{q}{4\pi\varepsilon_0}\frac{r}{R^3}$

$r > R \quad E = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$

$$W = \frac{\varepsilon_0}{2}\int_0^R (4\pi r^2 dr)\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{r^2}{R^6}$$

$$+ \frac{\varepsilon_0}{2}\int_R^\infty (4\pi r^2 dr)\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4}$$

$$= \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{4\pi}{R^6}\int_0^R dr\, r^4$$

$$+ \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 4\pi \int_R^\infty \frac{dr}{r^2}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2R^6}\right)\frac{R^5}{5} + \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2}\right)\frac{1}{R}$$

$$= \frac{q^2}{4\pi\varepsilon_0 R}\left(\frac{1}{10}+\frac{1}{2}\right) = \frac{3}{5}\frac{q^2}{4\pi\varepsilon_0 R}$$

c) $$W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\,\vec{E}\cdot d\vec{a}\right)$$

where V is a volume of radius a, S is a spherical surface of radius a.

Do the volume integral

$$\int_V E^2 d\tau = \int_0^R (4\pi r^2 dr)\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{r^2}{R^6}$$

$$+ \int_R^a (4\pi r^2 dr)\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4}$$

$$= \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{4\pi}{R^6}\int_0^R dr\, r^4 + \left(\frac{q}{4\pi\varepsilon_0}\right)^2 4\pi \int_R^a \frac{dr}{r^2}$$

So $$\frac{\varepsilon_0}{2}\int_V E^2 d\tau = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2R^6}\right)\frac{R^5}{5}$$

$$+ \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2}\left(\frac{1}{R} - \frac{1}{a}\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{10R} + \frac{1}{2R} - \frac{1}{2a}\right)$$

Do the surface integral

$$\frac{\varepsilon_0}{2}\oint_S V\vec{E}\cdot d\vec{a} = \left(\frac{q}{4\pi\varepsilon_0 a}\right)\left(\frac{q}{4\pi\varepsilon_0 a^2}\right)4\pi a^2\left(\frac{\varepsilon_0}{2}\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2a}\right)$$

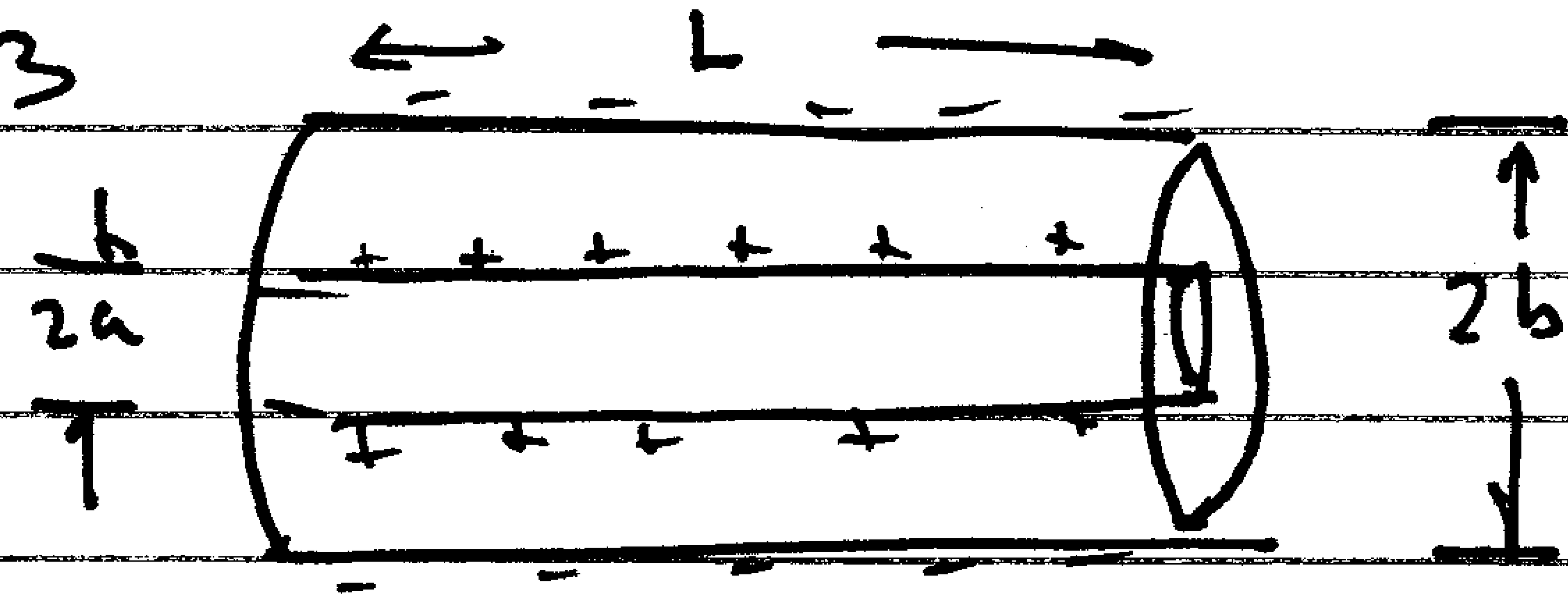
$$W = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{10R} + \frac{1}{2R} - \frac{1}{2a} + \frac{1}{2a}\right) = \frac{q^2}{4\pi\varepsilon_0 R}\left(\frac{3}{5}\right)$$

Note that as $a \to \infty$, the surface integral $\sim \frac{1}{2a} \to 0$

and the volume integral

$$\sim \frac{3}{5R} - \frac{1}{2a} \to \frac{3}{5R}$$

2.43



Put λ - charge on inner cylinder, $-\lambda$ on
length
the outer cylinder

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s} \qquad V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{l}$$

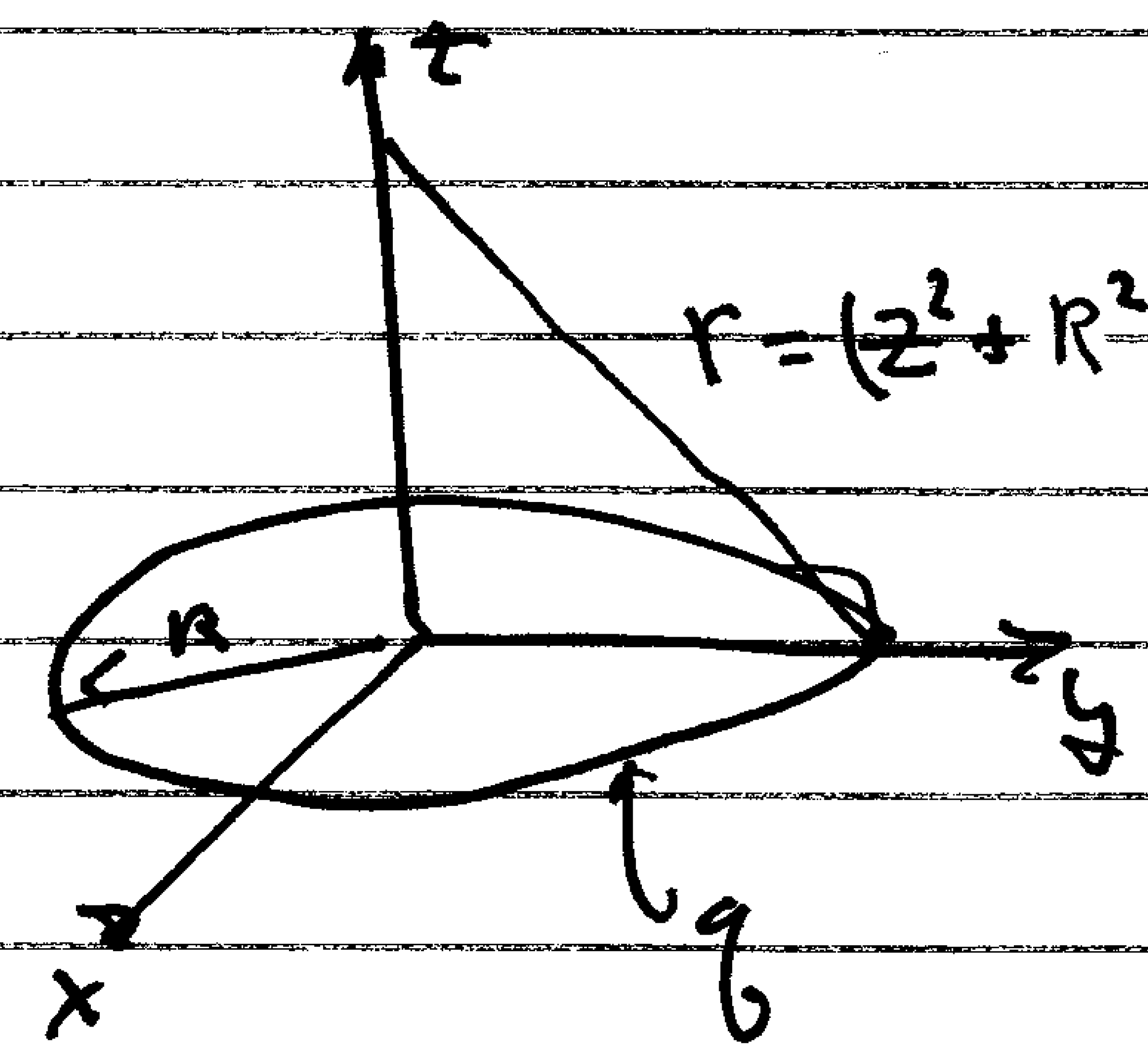
$$V_b - V_a = -\Delta V = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

$$\Delta V = V_a - V_b = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) \qquad \lambda = \frac{Q}{L}$$

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{Q}{L}\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{Q}{C}$$

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)} \quad \text{or} \quad \frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

3.28 We did this problem in lecture but it's worth going through it again



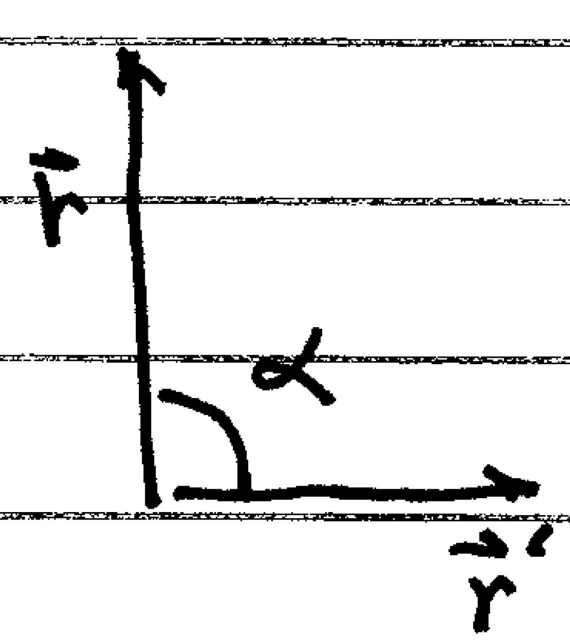
$$V(z) = \frac{q}{(z^2+R^2)^{1/2}}\left(\frac{1}{4\pi\epsilon_0}\right)$$

$z \geq R$ $$\frac{1}{(z^2+R^2)^{1/2}} = \frac{1}{z\left(1+\frac{R^2}{z^2}\right)^{1/2}}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{R}{z}\right)^n P_n(0)$$

using $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

with $r' = R$, $r = z$, $\alpha = \pi/2$



Note $P_n(0) = 0$ for n odd

$$z \geq R \quad V(z) = \frac{1}{4\pi\varepsilon_0} \frac{q}{z} \sum_{n=0}^{\infty} \left(\frac{R}{z}\right)^n P_n(0)$$

Let $z^{-n-1} \rightarrow r^{-n-1} P_n(\cos\theta)$

$$(z \geq R) \quad V(r,\theta) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n P_n(0)\, P_n(\cos\theta)$$

Now let $z \leq R$

$$\frac{1}{(R^2+z^2)^{1/2}} = \frac{1}{R\left(1+\frac{z^2}{R^2}\right)^{1/2}} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{z}{R}\right)^n P_n(0)$$

$$V(z) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{z}{R}\right)^n P_n(0)$$

Let $z^n \rightarrow r^n P_n(\cos\theta)$

$$(z \leq R) \quad V(r,\theta) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n P_n(0)\, P_n(\cos\theta)$$

Note: $P_0(0) = 1 \quad P_2(0) = -1/2 \quad P_4(0) = 3/8$

SP 1)

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta) \qquad r > a$$

Let $V(a,\theta) \equiv V_0(\theta)$

$$V(a,\theta) \equiv V_0(\theta) = \sum_{l=0}^{\infty} \frac{A_l}{a^{l+1}} P_l(\cos\theta)$$

Using the orthogonal relationship:

$$\int_0^{\pi} d\theta \sin\theta \, P_l(\cos\theta) \, P_{l'}(\cos\theta) = \frac{2}{2l+1} \delta_{l,l'}$$

we have

$$\int_0^{\pi} d\theta \, V_0(\theta) \sin\theta \, P_l(\cos\theta) = \frac{A_l}{a^{l+1}} \left(\frac{2}{2l+1}\right)$$

$$A_l = a^{l+1} \left(\frac{2l+1}{2}\right) \int_0^{\pi} d\theta \, V_0(\theta) \sin\theta \, P_l(\cos\theta)$$

$$V_0(\theta) = V_0 \quad 0 \le \theta \le \pi/2$$

$$V_0(\theta) = -V_0 \quad \pi/2 < \theta \le \pi$$

$$A_\ell = a^{\ell+1}\left(\frac{2\ell+1}{2}\right)\left[V_0\int_0^{\pi/2} d\theta \sin\theta \, P_\ell(\cos\theta) - V_0 \int_{\pi/2}^{\pi} d\theta \sin\theta \, P_\ell(\cos\theta)\right]$$

Let $x = \cos\theta \qquad dx = -\sin\theta \, d\theta$

$$A_\ell = a^{\ell+1}\left(\frac{2\ell+1}{2}\right) V_0 \left[(-)\int_1^0 dx \, P_\ell(x) - \int_0^{-1}(-) dx \, P_\ell(x)\right]$$

$$= a^{\ell+1}\left(\frac{2\ell+1}{2}\right) V_0 \left[\int_0^1 dx \, P_\ell(x) + \int_0^{-1} dx \, P_\ell(x)\right]$$

$$\int_0^{-1} dx \, P_\ell(x) = -\int_0^1 dx \, P_\ell(-x)$$

$$A_\ell = a^{\ell+1}\left(\frac{2\ell+1}{2}\right) V_0 \left[\int_0^1 dx \left(P_\ell(x) - P_\ell(-x)\right)\right]$$

$P_\ell(-x) = P_\ell(x)$ for ℓ even $\Rightarrow A_\ell = 0$ ℓ even

$P_\ell(-x) = -P_\ell(x)$ for ℓ odd

$$A_\ell = a^{\ell+1}\left(\frac{2\ell+1}{2}\right) V_0 (2) \int_0^1 dx \, P_\ell(x) \quad \ell \text{ odd}$$

$l=1 \qquad \int_0^1 dx\, P_l(x) = \int_0^1 dx\, x = \frac{1}{2}$

$l=3 \qquad \int_0^1 dx\, P_l(x) = \int_0^1 dx \left(\frac{5x^3-3x}{2}\right)$

$$= \frac{1}{2}\left(\frac{5}{4} - \frac{3}{2}\right) = -\frac{1}{8}$$

$l=5 \qquad \int_0^1 dx\, P_l(x) = \frac{1}{8}\int_0^1 dx\,(63x^5 - 70x^3 + 15x)$

$$= \frac{1}{8}\left(\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right) = \frac{1}{16}$$

So $\quad A_l = 2a^{l+1}\left(\frac{2l+1}{2}\right) V_0\, I_l$

$I_l = 0 \quad l$ even

$I_l = \frac{1}{2} \quad l=1$

$I_l = -\frac{1}{8} \quad l=3$

$I_l = \frac{1}{16} \quad l=5$ etc

$$V(r,\theta) = \sum_{l\ \text{odd}} \frac{a^{l+1}}{r^{l+1}}(2l+1)\, V_0\, I_l\, \frac{P_l(\cos\theta)}{r^{l+1}}$$

$$= V_0 \sum_{l\ \text{odd}} (2l+1)\, I_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$$

Now let $r < a$

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

But V must be continuous across the sphere

For $r > a$, we have

$$V_{out}(r,\theta) = V_0 \sum_{l\,odd} (2l+1) I_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$$

Let $r = a$

$$V_0 \sum_{l\,odd} (2l+1) I_l P_l(\cos\theta) = \sum_l A_l a^l P_l(\cos\theta)$$

$$\Rightarrow A_l = \frac{V_0 (2l+1) I_l}{a^l} \qquad l \text{ odd}$$

And $V_{in}(r,\theta) = V_0 \sum_{l\,odd} (2l+1) I_l \left(\frac{r}{a}\right)^l P_l(\cos\theta)$